ALGEBRAICKÉ VARIETY

§1 : Úvod

- pracujeme v n-rozmernom Euklidovskom (afinnom) pries[illegible] E^n nad poľom $\mathbb{K}$ - pole je algebraicky uzavreté (každý polyn[illegible] nad týmto poľom má v poli koreň)

$\mathbb{K}[x_1, \dots, x_n]$ - okruh polynómov n-neurčitých

Def 1.1 : Algebraická varieta v E_n^n - nazveme ňou množinu $\mathbb{V}$ definovanú : $\mathbb{V} = \{X = (x_1 \dots x_n) \in E^n ; f_1(x_1 \dots x_n) = \dots = f_s(x_1, \dots x_n)\}$
$f_i \in k[x_1, \dots x_n]$ pevné $s \in \mathbb{N}$

PR 1: každá lineárna varieta v E_n je algebraická varieta

2: rovina v E_3 $\quad x - y + z - 1 = 0$

priamka v E_3 $\quad x + y - 1 = 0$
$\quad x - z + 2 = 0$

bod v E_3 $\quad x - 1 = 0 \quad$ bod $[1,1,1]$ - jedno ri[illegible]
$\quad y - 1 = 0$
$\quad z = 1 = 0$

nech $\mathbb{V}$ je AV daná rovnicami $f_1 = \dots = f_s = 0$

*Definujeme : $k[x_1, \dots, x_n] \supset \mathbb{I} = \{g_1 f_1 + \dots + g_s f_s ; \ g_i \in k[x_1, \dots x_n]\}$

[illegible] do ALGEBRY

Def 1.2 : Ideálom v okruhu $\mathbb{R}$ (komutatívny, noetherovský) nazveme neprázdnu podmnožinu $I \subset R$, tj. spĺňa :

1. $\forall a, b \in I : a - b \in I$ $\quad$ I podgrupa [illegible]
2. $\forall a \in I, \forall r \in \mathbb{R} : r \cdot a \in \mathbb{I}$

PR : $(\mathbb{Z}, +) \supset (n \cdot \mathbb{Z})$ - ideály
podgrupy sú $\mathbb{N}$

* tak def. I-čko označ. $=: (f_1, \dots f_s) \cdot k[x_1, \dots x_n]$ - ideál v $k[x_1, \dots x_n]$

Def 1.3 : nech $I = (f_1, \dots f_r) . k[x_1, \dots x_n]$
$E_n \supset \mathcal{V}(\mathbb{I}) = \{X = (x_1, \dots x_n) \in E_n ; f(x_1, \dots x_n) = 0 \ \forall f \in \mathbb{I}\}$
$\uparrow$ množina nulových bodov ideálu I.

$(f_1, \dots f_s)$ generátory I $\qquad x \in X : f_1 = \dots = f_r = 0$
[illegible] každý [illegible]

Algebraická varieta je množinou všetkých nulových bodov ideálu v $k[x_1, .. x_n]$ $V(I)$

každý ideál má konečnú bázu

$k[x_1, .. x_n] \supseteq \longrightarrow E_n \supseteq \rightarrow k[x_1, .. x_n]$ [illegible] - variet.

$I \longmapsto V(I)$

ideál ASOCIOVANÝ IDEÁL s varietou V

$V \longmapsto J(V) =: \{ f \in k[x_1, .. x_n] ; f(x_1, .. x_n) = 0 \ \forall (x_1, .. x_n) \in W \}$

Platí $I \subset J(V)$. ale obrátene nie

PR: $I = ((x - y + 1)^2 \cdot k[x, y] \Rightarrow V(I)$

$\Rightarrow J(V(I)) = (x - y + 1) \cdot k[x, y]$

Def 1.4: nech $I \subset R$ je ideál

radikál

$\text{rad } I = \{ r \in R ; r^g = I \text{ pre } \text{určité } g \in N \}$

$\cdot I \subset \text{rad } I$

Platí: $J(V(I)) = \text{rad } (I)$ Hilbertova veta radikál

Def 1.5 Nech $I, J \subset R$ z ľubovol. okruhu sú ideály.

definujem operácie

$I + J =: \{ a + b ; a \in I, b \in J \}$

$I \cap J =: \{ a \in R ; a \in I, a \in J \}$

Veta 1.6: $I, J \in k[x_1, .. x_n]$ $V, W \subset E_n$

1. ak J je časťou J $J \subset J \Rightarrow V(I) \supset V(J)$

2. $V \subset W \Rightarrow J(V) \supset J(W)$

Dôkaz:

3. $V(I + J) = V(I) \cap V(J)$

$I(f_1, .. f_s)$ $I + J = (f_1, \ f_k ; f_{k+1}, .. f_r)$ [illegible]

$J(f_1, \ f_k)$

4. $V(I \cap J) = V(I) \cup V(J)$

Dôkaz 3: $I \subset I + J \Rightarrow V(I + J) \subset V(I)$

$J \subset I + J \Rightarrow V(I + J) \subset V(J)$

„$\supseteq$" $x \in V(I) \cap V(J) \rightarrow f(x) = 0 \ \forall f \in I$ $(f + g)(x) = 0 \ \forall f + g \in [I]$

$g(x) = 0 \ \forall g \in J$

④ $\supseteq$ zrejmá

$$I \cap J \subset I \Rightarrow V(I) \supset V(I \cap J)$$

$\subseteq$ def: $I \cdot J = \{a \cdot b,\ a \in I,\ b \in J\}$

$$I \cdot J \subset I \cap J \subset \begin{smallmatrix} I \\ J \end{smallmatrix}$$

$$V(I \cdot J) = V(I) \cup V(J)$$

$x \in V(I \cdot J) \quad x \notin V(I)$

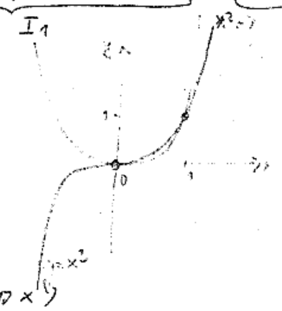
$(f \cdot g)(x) = 0 \quad \forall f = J \quad g \in J \quad f(x) \neq 0 \quad (f \cdot g)(x) = f(x) \cdot g(x) = 0 \to g(x) = 0$

$\Rightarrow x \in V(J)$

Príklady: $k[X,Y] \longleftrightarrow E_2$

$$I = (Y - X^2,\ Y - X^3) \cdot k[X,Y] = \underbrace{(Y-X^2) \cdot k[X,Y]}_{I_1} + \underbrace{(Y-X^3) \cdot k[X,Y]}_{I_2}$$

$$V(I) = V(I_1) \cap V(I_2)$$



$y = x^2$
$y = x^3$

$x^3 = x^2$
$x^3 - x^2 = 0$
$x \cdot (x-1) = 0$
$0 \vee 1$

- v bode $[0,0]$ - majú spoločnú dotyčnicu $(0x)$

$[1,1]$ - nemajú

$$\begin{tikzcd}
k[x_1, \dots x_n] & E^n & k[x_1, \dots x_n] \\
I \arrow[r, squiggly] & V(I) = \{(x) \in E_n;\ f(x) = 0,\ \forall f \in I\} & \\
& V \arrow[r, squiggly] & J(V) = \{f \in k[x_1 \dots x_n];\ f(x) = 0,\ \forall x \in V\}
\end{tikzcd}$$

(with $k[x_1..x_n] \supset I$, $E^n \supset V(I)$, $V(I) = V$, $k[x_1 \dots x_n] \supset$)

$\parallel$ platí

$\mathrm{rad}(I) = \{f \in k[x_1 \dots x_n],\ \exists \varsigma \in N,\ f^{\varsigma} \in I\}$

$V(I) = V(\mathrm{rad}\ I)$

$V(I) = V(J) \Leftrightarrow I = J$!

§2 PRIMÁRNE ZÁKLADY IDEÁLOV, IREDUCIBILNÉ ROZKLADY VARIET.

Def: $X \subset E_n$ sa nazýva ireducibilná, ak spĺňa:
(varieta)

$X = X_1 \cup X_2 \quad X_1, X_2$ - alg. variety $\Rightarrow X = X_1 \vee X = X_2$

Def: $I \subset (k[x_1, \dots x_n])\ \mathbb{R}$, ideál nazveme prvoideálom, ak platí

$$a \cdot b \in I,\ a \notin I \Rightarrow b \in I$$

PR: $Z \supset Z(n)$ - prvoideál $\Leftrightarrow$ n prvočíslo

Veta: Varieta $X \in E_n$ je ireducibilná $\Leftrightarrow J(X)$ je prvoideál

Dôkaz $\boxed{\Rightarrow}$ X je ired. a $\boxed{f \cdot g \in J(X)}$ $\qquad V(f) = \{(x) \in E_n,\ f(x) = 0\}$

$$X = (\underset{\cap\, X}{X \cap V(f)}) \cup (\underset{\cap\, X}{X \cap V(g)})$$

$\supset$

$\subseteq (x) \in X \Rightarrow (f \cdot g)(x) = 0 \Rightarrow f(x) = 0 \vee g(x) = 0 \Rightarrow (x) \in V(f) \cup (x) \in V(g) \Rightarrow (x) \in X \cap V$

$\vee (x) \in X \cup V(g)$

$\Rightarrow$ $\mathbb{X} = \mathbb{X} \cap \mathcal{V}(f) \vee \mathbb{X} = \mathbb{X} \cap \mathcal{V}(g) \Rightarrow \mathbb{X} \subset \mathcal{V}(f) \vee \mathbb{X} \subset \mathcal{V}(g) \Rightarrow \underline{f \in \mathcal{I}(\mathbb{X}) \vee g} \in \mathcal{I}(\mathbb{X}$

$\mathcal{I}(\mathbb{X})$ je prvoideál

$\boxed{\Leftarrow}$ Nech $\mathcal{I}(\mathbb{X})$ je prvoideál a $\mathbb{X}$ je reducibilné

$X = \mathbb{X}_1 \cup \mathbb{X}$, $\mathbb{X}_1 \subsetneqq \mathbb{X}$ $\quad \mathbb{X}_2 \subsetneqq \mathbb{X}$

$\mathcal{I}(X) = \mathcal{I}(\mathbb{X}_1) \cap \mathcal{I}(\mathbb{X}_2) \Rightarrow \mathcal{I}(\mathbb{X}) \subsetneqq \mathcal{I}(\mathbb{X}_1) \quad \mathcal{I}(\mathbb{X}) \subseteq \mathcal{I}(\mathbb{X}_2)$

$$\left.\begin{array}{l}\exists f \in \mathcal{I}(\mathbb{X}_1) / \mathcal{I}(\mathbb{X}) \\ \exists g \in \mathcal{I}(\mathbb{X}_2) / \mathcal{I}(\mathbb{X})\end{array}\right\} \Rightarrow f \cdot g \in \mathcal{I}(\mathbb{X}_1) \cap \mathcal{I}(\mathbb{X}_2) = \mathcal{I}(\mathbb{X}), \text{ ale } f \notin \mathcal{I}(\mathbb{X}), g \notin \mathcal{I}(\mathbb{X}$$

→ nie je prvoideál SPOR ↯

Def: $\mathbb{I} \subset \mathbb{R}$ nazveme primárny, ak spĺňa:

$f \cdot g \in \mathbb{I} \wedge f \in \mathbb{I} \Rightarrow g^\rho \in \mathbb{I}$ pre určité $\rho \in \mathbb{N}$

Pr: $\mathbb{Z} \supset (p^\rho)\mathbb{Z}$ - mocnina prvočísla

Veta: Radikál primárneho ideálu je prvoideál

$\underset{\uparrow \text{primárny}}{\mathfrak{q}}$ - q $\quad \underset{\uparrow \text{rád}(\mathfrak{q})}{\mathfrak{p}}$ - p $\quad$ Schwabachove

Dôkaz: $\mathfrak{q}$ primárny. Nech $f, g \in \mathrm{rad}(\mathfrak{q})$, $f \notin \mathrm{rad}(\mathfrak{q}) \Rightarrow$

$\Rightarrow \underset{\downarrow f^\rho \cdot g^\rho}{(f \cdot g)^\rho} \in \mathfrak{q}$, $f^\rho \notin \mathfrak{q} \Rightarrow (g^\rho)^\lambda \in \mathfrak{q} \Rightarrow \underline{\underline{g \in \mathrm{rad}(\mathfrak{q})}}$

$\underline{\underline{\sqrt{\mathfrak{q}} \text{ je } \mathfrak{p}\text{-primárny}}}$

Def: Ideál $\mathfrak{a} \subset \mathbb{R}$ nazveme ireducibilný, ak spĺňa:

$\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2 \Rightarrow \mathfrak{a} = \mathfrak{a}_1 \vee \mathfrak{a} = \mathfrak{a}_2$

Uvedomme si: Každá alg. varieta je zjednotením konečného # ireducibilných alg vari

$\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$

$\quad \| $

$\quad \mathbb{X}_{11} \cup \mathbb{X}_{12}$

$\quad \|$

$\quad \mathbb{X}_{111} \cup \mathbb{X}_{112} \quad \dots \mathbb{X}_{1\dots1} \cup \mathbb{X}_{1\dots2}$

$\mathbb{X} \dots \subsetneqq \mathbb{X}_{11} \subsetneqq \mathbb{X}_1 \subsetneqq \mathbb{X}$

↳ konečné, lebo rozmer nejde do záporných č.

→ posledná varieta je bod

Veta: Každý ideál je prienikom konečného počtu ireducibilných

Dôkaz: $\mathfrak{a} = \mathfrak{a}_1 \cap \underline{\mathfrak{a}_2}$ v $\mathbb{R}$ → noetherovský

$\quad \| $

$\quad \mathfrak{a}_{11} \cap \underline{\mathfrak{a}_{12}} \qquad \mathfrak{a} \subsetneqq \mathfrak{a}_1 \subsetneqq \mathfrak{a}_{11} \subsetneqq \mathfrak{a}_{111} \subsetneqq \dots$ končí lebo $\mathbb{R}$ je noetherovský

$\quad \| $

$\quad \mathfrak{a}_{111} \cap \mathfrak{a}_{112}$

$\quad \vdots$

Veta: Každý ireducibilný ideál je primárny

Dôkaz: $\mathfrak{a}$ - ired, $f \cdot g \in \mathfrak{a}$, $f \notin \mathfrak{a}$

ideál: $(\mathfrak{a} : g) \subset (\mathfrak{a} : g^2) \subset \dots \subset (\mathfrak{a} : g^r) = (\mathfrak{a} : g^{r+1})$

lebo noetherovský

$\quad \|$

$\{r \in \mathbb{R}; r \cdot g \in \mathfrak{a}\}$ dokážem $\quad (\underset{\cup\, \mathfrak{a}}{\mathfrak{a} + (g^r)}) \cap (\underset{\cup\, \mathfrak{a}}{\mathfrak{a} + (f)}) = \mathfrak{a}$

$\supseteq$

$\subseteq \ni x \Rightarrow x = \underset{\in \mathfrak{a}}{a} + u g^r = \underset{\in \mathfrak{a}}{a'} + v f \quad / g$

$g \cdot x = g \cdot \underset{\in \mathfrak{a}}{a} + u g^{r+1} = \underset{\in \mathfrak{a}}{a' g} + v \underset{\in \mathfrak{a}}{(f \cdot g)}$

$\Rightarrow u \cdot g^{r+1} \in \mathfrak{a} \Rightarrow u g^r \in \mathfrak{a} \Rightarrow x \in \mathfrak{a}$

$(\mathfrak{a} + (g^r)) \cap (\mathfrak{a} + (f)) = \mathfrak{a} \Rightarrow \mathfrak{a} = \mathfrak{a} + g^r \Rightarrow g^r \in \mathfrak{a}$

Dôsledok: Každý ideál v $\mathbb{R}$ je prienikom konečného počtu primárnych ideálov

$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \mathfrak{q}_s$

$\mathrm{rad}(\mathfrak{a}) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_s$

$\mathrm{rad}(\mathfrak{a} \cap \mathfrak{b}) \underset{\subseteq}{=} \mathrm{rad}(\mathfrak{a}) \cap \mathrm{rad}(\mathfrak{b})$

1.

2. $f \in \mathrm{rad}(\mathfrak{a} \cap \mathfrak{b}) \Rightarrow f^{\varrho} \in \mathfrak{a},\ f^{\lambda} \in \mathfrak{b} \Rightarrow f^{\max\{\varrho,\lambda\}} \in \mathfrak{a} \cap \mathfrak{b} \Rightarrow f \in \mathrm{rad}(\mathfrak{a}) \cap \mathrm{rad}(\mathfrak{b})$

$$V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a})) = \underbrace{\overset{V(\mathfrak{q}_1)}{\overset{\shortparallel}{V(\mathfrak{p}_1)}} \cup \overset{V(\mathfrak{q}_2)}{\overset{\shortparallel}{V(\mathfrak{p}_2)}} \cup \ldots \cup \overset{V(\mathfrak{q}_s)}{\overset{\shortparallel}{V(\mathfrak{p}_s)}}}_{\text{ireducibilné}}$$

Pr: $\mathfrak{a} = (x^2+y^2-1,\ x\cdot y) = (x^2+y^2-1,\ x) \cap (x^2+y^2-1,\ y) = (\underset{(y-1)(y+1)}{y^2-1}, x) \cap (\underset{(x-1)(x}{x^2-1}, y$

$\overset{\cap}{k}[x,y] = (y-1,x) \cap (y+1,x) \cap (x-1,y) \cap (x+1,y)$

prvoideály
$\subseteq$

$(0,1)$ $(0,-1)$ $(1,0)$ $(-1,0)$

Rozmer ALG. VARIETY, ROZMER IDEÁLU

Nech $X \subset E^n$ je ireducibilná alg. varieta

$X = X_s \supsetneq X_{s_1} \supsetneq X_{s_2} \supsetneq \ldots \underset{\hookrightarrow \text{krivka}}{X_1} \supsetneq \overset{\text{bod}}{X_0} \supsetneq \emptyset$ - ireduc. alg. variety
- maximálny reťazec

Def: $s := \dim X$ s - rozmer variety X

$$X = X_s \supsetneq X_{s-1} \supsetneq \ldots \supsetneq X_1 \supsetneq X_0 \supsetneq \emptyset$$

$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow \qquad \nwarrow\!\!\searrow$

asociovaný prvoideál bodu so súr. $a_1, \ldots, a_n$

$$\underset{\text{asociovaný prvoideál}}{\to} I(X_s) \subsetneq I(X_{s-1}) \subsetneq \ldots \subsetneq I(X_1) \subsetneq I(X_0) \subseteq k[x_1, \ldots, x_n]$$

$\shortparallel$

$(x_1-a_1, \ldots, x_n-a_n)$ maximál ~~ideál~~ prvoideál

Def: Nech $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ je prvoideál a $\mathfrak{p} = \mathfrak{p}_s \subsetneq \mathfrak{p}_{s-1} \subsetneq \ldots \subsetneq \mathfrak{p}_0 \subsetneq \mathfrak{m}$ - max ~~id~~ ideál
je maxim. reťazec prvoideálov

$s := \dim \mathfrak{p}$ s - rozmer prvoideálu $\mathfrak{p}$

Veta: $\dim X = \dim I(X)$, X - ired.

Nech $X = X_1 \cup X_2 \cup \ldots \cup X_z$ $\forall$ ireducibilné

Def: $\dim X := \max\{\dim X_1\}$

Platí $\dim I(X) = \max\{\dim I(X_i)\}$

Nech $\dim X = d$

$$\Downarrow$$

$$X = {}^{d}X_1 \cup \ldots \cup {}^{d}X_r \cup {}^{d-1}X_1 \cup \ldots \cup {}^{d-1}X_t \cup {}^{0}X_1 \cup \ldots \cup {}^{0}X_s$$

$$I(X) = I({}^{d}X_1) \cap \ldots \qquad \cap I(X_s)$$

$$\mathfrak{a} = {}^{d}\mathfrak{q} \cap \ldots \cap {}^{d}\mathfrak{q}_r \cap {}^{d-1}\mathfrak{q} \cap \ldots \cap {}^{d-1}\mathfrak{q}_t \cap \ldots$$

$$V(\mathfrak{a}) = V({}^{d}\mathfrak{q}) \cup \ldots \cup V({}^{d-1}\mathfrak{q}_t) \cup \ldots$$

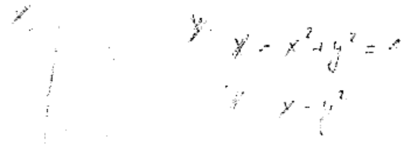
$\mathfrak{a} \in k[x_1, \dots x_n]$

$\dim \mathfrak{a} = 0 \Rightarrow \mathfrak{a} = {}^{c}\mathfrak{q}_1 \cap \dots \cap {}^{c}\mathfrak{q}_s$

$V(\mathfrak{a}) = c_1 \cup \dots \cup c_s$ $-$ c_i - body

BEZOUTOVA VETA: v $E^2 \overset{\to \text{algebr. uzavreté}}{\Leftrightarrow} k[x,y]$

$X: F = 0$
$Y: G = 0$ $\}$ krivky $\deg F = m$, $\deg G = n$

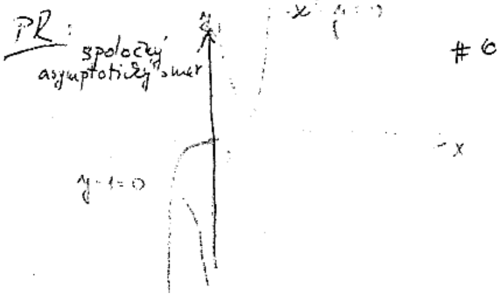


$\# X \cap Y = m \cdot n$, pričom každý bod sa počíta s príslušnou <u>násobnosťou</u>
(medzi spoločné body spočítame aj spoločné asymptotické smery)

PR: $x^2+y^2=1$
$y = x^2$

4 spoločné body

PR:



$\# 0$

PR: $\mathfrak{a} = (x^2-y^2-4, x^2y^2) = \underset{\underset{(x-2,y)}{\cap}}{(x-2,y^2)} \cap \underset{\underset{(x-2,y)}{\cap}}{(x+2,y^2)} \cap$
$\underset{\underset{(y-2,x)}{\cap}}{(y-2,x^2)} \cap \underset{\underset{(y+2,x)}{\cap}}{(y+2,x^2)}$

$V(\mathfrak{a}) = \underset{2x}{(2,0)} \cup \underset{2x}{(-2,0)} \cup \underset{2x}{(0,2)} \cup \underset{2x}{(0,-2)}$ $\# 8$

USPORIADANIA V KRUHU POLYNÓMOV

$k[x_1, \dots x_n] =: R$; $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}$ - monóm - polynóm, kde nie sú súčty, iba súčiny

$\alpha = [\alpha_1, \dots, \alpha_n]$

Def: Usporiadaním na monómoch R rozumieme reláciu $>$, k.t. spĺňa

1. $X^\alpha > X^\beta \Rightarrow X^\beta \not> X^\alpha$ - antisymetria
2. $X^\alpha > X^\beta$, $X^\beta > X^\gamma \Rightarrow X^\alpha > X^\gamma$ tranzitívnosť
3. z trojice $X^\alpha > X^\beta$, $X^\beta > X^\alpha$, $X^\beta = X^\alpha$ platí práve 1

$\}$ > lineárne usporiadanie

4.) $X^\alpha > X^\beta \Rightarrow X^\alpha \cdot X^\gamma > X^\beta \cdot X^\gamma$ - kompatibilnosť s násobením

<u>Monomiálne</u> usporiadanie v $k[x_1, \dots x_n]$ $x_1 > x_2 > \dots > x_n$

1 lineárne
2. kompatibilné s násobením
3. $x_i > 1$ $\forall i = 1, 2 \dots$

~~Def~~: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$

$\alpha - \beta = (0, \dots, 0, \underset{\neq 0}{\alpha_r - \beta_r}, \dots \underset{\neq}{\alpha_s - \beta_s}, 0, \dots 0)$

Def: $x^\alpha \underset{lex}{>} x^\beta$ v <u>lexikografickom</u> usporiadaní $\overset{df}{\Longleftrightarrow}$ $\alpha_r - \beta_r > 0$

PR: $k[x,y,z]$

$x^3y,\ xy^2z,\ xyz^2,\ y^3,\ z^4$ uspriadanie: $x^3y > xy^2z > xyz^2 > z^4$

[illegible]

porovnám sa na x, kde x predchádza so p väčšie, až x má rovnaké porovnám na y

a podmienky 3

; Keby $x_i < 1$ $x_1^2 > ?$

Gradované lexikografické usp.: (dĺžka monómu) je monomiálne $<\begin{matrix}Lex\\ Glex\end{matrix}$

Gradované usp. je monomiálne !

$|\alpha| = \sum \alpha_i$

napr. ide monómy, kt. majú rovnakú dĺžku

$x^\alpha > x^\beta$ v Glex $\Longleftrightarrow$ $|\alpha| > |\beta|$ alebo $|\alpha| = |\beta|$ a $\alpha_r - \beta_r > 0$

v príklade: $x^3y > xy^2z > xyz^3 > z^4 > y^3$

$>$ je monomiálne na $k[x_1, \dots x_n]$

$f \in k[x_1, \dots x_n]$

$\| \sum c_\alpha x^\alpha$ Nech x^{α_0} je vedúci monóm v f $\longrightarrow$ $\boxed{Lt(f) := c_{\alpha_0} x^{\alpha_0}}$

(každý polynóm má vedúci člen !)

Ideál v okruhu $I = (f_1, \dots f_s) \subset k[x_1, \dots x_n]$

Definujem si vedúci ideál ideálu I: [illegible] generovaný monómami

$$Lt(I) = \{Lt(f);\ f \in I\} \supset (Lt(f_1), \dots Lt(f_s))$$

PR: $I = (\overset{\text{vedúci č.}}{\underline{x^2}} - y,\ \overset{\text{vedúci č.}}{\underline{x}} - y^3)$ $(x, x^2) = (x) \in Lt(f_i)$

uspor. Lex

$\Downarrow$ $\Downarrow$ násobím x

$x^2 - y,\ x^2 - xy^3 \to$ odčítam

$xy^3 - y$ | [illegible] y^2 $xy^2 - y^6$ odčítam

$y^6 - y$

$y^6 \in Lt(f)$

$\subset$

DELIACI ALGORITMUS v $k[x_1, \dots x_n]$

Pr. $f = x^3 - 2x + 1$ $f : g = (x^3 - 2x + 1) : (x^2 + x) = x - 1$ $\Rightarrow f = g(x-1) + (-x+1)$

určené jednoznačne

$g = x^2 + \cancel{x}$

$$\begin{array}{l}
-(x^3 + x^2)\\
\hline
-x^2 - 2x + 1\\
-(-x^2 - x)\\
\hline
-x + 1
\end{array}$$

vo viac premenných: delíme [illegible] polynómami

PR: $k[x,y]$

$\Gamma = \{\underbrace{\underline{x}^2 - y}_{F_1},\ \overbrace{\underline{x} - y^2}^{F_2}\}$

$f \overset{Lex}{=} \underline{x}^4 - x^3y + xy - 1$

budem ho deliť

$(x^4 - x^3y + xy - 1) : (x^2 - y) = x^2 - xy + y$

$$\begin{array}{l}
-(x^4 - x^2y)\\
\hline
-x^3y + x^2y + xy - 1\\
-(-x^3y + xy^2)\\
\hline
x^2y + xy - xy^2 - 1\\
-(x^2y - y^2)\\
\hline
xy - xy^2 + y^2 - 1
\end{array}$$

$*$ $-xy^2 + xy + y^2 - 1 : (x - y^2) = -y^2 + y$

$$\begin{array}{l}
-(-xy^2 + y^4)\\
\hline
xy - y^4 + y^2 - 1\\
-(xy - y^3)\\
\hline
-y^4 + y^3 + y - 1
\end{array}$$

$\Rightarrow f = (x^2 - xy + y)F_1 + \overset{*}{(}-xy^2 + xy + y^2 - 1) =$

$= (x^2 - xy + y)F_1 + (-y^2 + y)F_2 + (-y^4 + y^3 + y - 1$

PR: $k[x,y,z]$

$\Gamma\{\underset{F_1}{\underline{x^2}-y}, \underset{F_2}{\underline{x}-z}, \underset{F_3}{y-z^2}\}$

$f = x^2y - xy^2 + xz - z^2$

$x^2y - xy^2 + xz - z^2 : (x^2 - y) = y$

$-(x^2y - y^2)$

$-xy^2 + xz + y^2 - z^2 : (x - z) = -y^2 + z$

$-(-xy^2 + y^2z)$

$xz - y^2z + y^2 - z^2$

$-(xz - z^2)$

$-y^2z + y^2 : (y - z^2) = -yz + y - z^3 + z^2$

$-(-y^2z + yz^3)$

$y^2 - yz^3$

$-(y^2 - yz^2)$

$-yz^3 + yz^2$

$-(-yz^3 + z^5)$

$yz^2 - z^5$

$-(yz^2 - z^4)$

$-z^5 + z^4$

$$f = y \cdot F_1 + (-y^2 + z)F_2 + (-yz + y - z^3 + z^2)F_3 + (-z^5 + z^4)$$

Veta: Deliaci algoritmus v $k[x_1, .. x_n]$
nech $F_1, .. F_s \in k[x_1, .. x_n]$ $f \in k[x_1, .. x_n] \Rightarrow \exists g_1, .. , g_s,$
$r \in k[x_1, .. x_n]$ $f = g_1F_1 + g_2F_2 + .. + g_3F_3 + r$
pričom (pre každý monóm z r) monómy z r nie sú deliteľné
monómmi $Lt(F_1), \dots, Lt(F_s)$

$\Gamma = \{F_1, .. F_s\}$ r - zvyšok po delení polynómu f množinou Γ

$r := \overline{f}^{\Gamma}$

Gröbnerove bázy GB

Def: Bázu $F_1, .. F_s$ ideálu I nazveme Gröbnerovou bázou ak [illegible]

$Lt(I) = (\underset{\text{monóm}}{Lt(F_1)}, .. \underset{\text{monóm}}{Lt(F_s)})$ = [illegible] I, iba monómy

Je generovaný $\{F_1, ... F_s\}$ [illegible]

$I = (F_1, ... F_s)$

Def: Nech $f, g \in k[x_1, .. x_n]$ S-polynómom polynómov f, g rozumieme polynóm $S(f,g) = \frac{x^\gamma}{Lt(f)} f - \frac{x^\gamma}{Lt(g)} g$ pričom x^γ je najmenší spoločný násobok

[illegible] x^α a x^β, $Lt(f)$ - niečo$\cdot x^\alpha$ → [illegible] ako hľadať GB

PR: $f = x^2y - y^3$
$g = xy^2 - xy$

$$S(f,g) = \frac{x^2y^2}{x^2y}(x^2y - y^3) - \frac{x^2y^2}{xy^2}(xy^2 - xy) = -y^4 + x^2y = x^2y - y^4$$

pojednaný [illegible], spájaním, [illegible] $= 0$

Veta: Báza $I = (F_1, \dots F_s)$ je Gröbnerova $\iff$ keď $\overline{S(F_i, F_j)}^{\Gamma}$, $\Gamma = \{F_1, \dots F_s\}$

$$\boxed{\overline{S(F_i, F_j)}^{\Gamma} = 0}$$

PR: $(\underbrace{x - y^2}_{F}, \underbrace{y^3 - y}_{G})$ lex

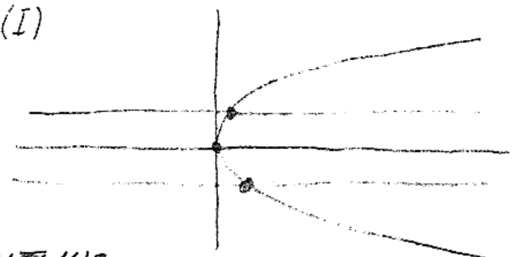
$$S(F,G) = \frac{xy^3}{x}F - \frac{xy^3}{y^3}G = y^3x - y^5 - (y^3x - xy) =$$

$= -y^5 + xy : y^3 - y = y^2$
$-(y^5 - y^3)$
$xy + y^3 : x - y^2 = y$
$-(xy - y^3)$
0

$-y^5 + xy : y^3 - y = -y^2 - 1$
$-(-y^5 + y^3)$
$xy - y^3$
$-(xy - y^3)$

$xy - y : x - y^2 = y$
$-xy + y^3$
$y^3 - y : y^3 - y = 1$
0

$V(I)$



$LI(I) = (x, y^3)$

- má 3 body

BUCHBERGEROV ALGORITMUS

1. $I = (f_1, \dots f_s)$ $\Gamma_1 = \{f_1, \dots f_s\}$

$\overline{S(f_i, f_j)}^{\Gamma_1}$ Označme $g_1 \dots g_l$ všetky nenulové zvyšky

sú zbytočné lebo je to [illegible] ideál

2. $I = (f_1, \dots f_s, g_1, \dots g_l)$ $\Gamma_2 = \{f_1, \dots f_s, g_1, \dots g_l\}$

$\overline{S(f_i, f_j)}^{\Gamma_2} = 0$ $\overline{S(f_i, g_j)}^{\Gamma_2}$ $\overline{S(g_i, g_j)}^{\Gamma_2}$

Nech $h_1 \dots h_k$ sú všetky nenulové zvyšky

⋮

po niekoľkých krokoch sú všetky bez zvyšku $(\forall\, \overline{S} = 0)$

PR: $I = (\underbrace{x^2 - y^3}_{f}, \underbrace{xy - y^2}_{g}) = \Gamma_1$ lex

$$S(f,g) = \frac{x^2y}{x^2}f - \frac{x^2y}{xy}g = -y^4 + x\cdot y^2$$

$-y^4 + xy^2 : xy - y^2 = y$
$(-y^3 + xy^2)$
$-y^4 + y^3$

$\overline{S(f,g)}^{\Gamma_1} = y^4 - y^3$

$I = (\underbrace{x^2 - y^3}_{f}, \underbrace{xy - y^2}_{g}, \underbrace{y^4 - y^3}_{h})$ Γ_2

$\overline{S(f,g)}^{\Gamma_2} = 0$

$$S(f,h) = \frac{x^2y^4}{x^2}f - \frac{x^2y^4}{y^4}h = -y^7 + x^2y^3 : y^4 - y^3 = -y^3$$

$-(-y^7 + y^6)$
$x^2y^3 - y^6 : x^2 - y^3 = y^3$
$-(x^2y^3 - y^6)$
0

$\overline{S(f,h)}^{\Gamma_2} = 0$

$$S(g,h) = \frac{xy^4}{xy}g - \frac{xy^4}{y^4}h = -y^5 + xy^3 : y^4 - y^3 = -y$$

$-(-y^5 + y^4)$
$xy^3 - y^4 : xy - y^2 = y^2$
$-(xy^3 - y^4)$
0

$\overline{S(g,h)}^{\Gamma_2} = 0$

$I = (x^2 - y^3, xy - y^2, y^4 - y^3)$ — GRÖBNEROVA BÁZA

$LI(I) = (x^2, xy, y^4) \longrightarrow 5$

$S(f,g) = -x^3+y^4 : y^3-x^2 = y$

$-(y^4-x^2y)$

$x^2y - x^3 : xy-y^2 = x$

$\underline{-(x^2y-xy^2)}$

$-x^3+xy^2 : xy-y^2 = y$

$\underline{-(xy^2-y^3)}$

$-x^3+y^3 : y^3-x^2 = 1$

$\underline{-(-x^2+y^3)}$

$\underline{\underline{-x^3+x^2}}$

$I = (y^3-x^2, xy-y^2, -x^3+x^2)$

$\overline{S(f,g)}^{\Gamma_2} = 0$

$\overline{S(f,h)}^{\Gamma_2} = 0$ z lemy $(y^3, -x^3) = 1$

<u>PR</u>

$S(g,h) = \frac{x^3y}{xy}g - \frac{x^3y}{-x^3}h$

$-x^2y^2+x^2y : xy-y^2 = -xy$

$\underline{-(-x^2y^2+xy^3)}$

$x^2y-xy^3 : xy-y^2 = x$

$\underline{-(x^2y-xy^2)}$

$-xy^3+xy^2 : xy-y^2 = y$

$\underline{-(-y^3+xy^2)}$

$-xy^3+y^3 : y^3-x^2 = 1$

$\underline{-(y^3-x^2)}$

$x^2-xy^3 : y^3-x^2 = -x$

$\underline{-(x^3-xy^3)}$

$x^2-x^3 : (-x^3+x^2) = 1$

$\overline{S(g,h)}^{\Gamma_2} = 0$

$Ll(I) = (x^3, xy, y^3)$

$\downarrow$

Γ

LEMA. $\{f,g\} = \Gamma^{\phi}$ $(Ll(f), Ll(g)) = 1$ - nsd $\Rightarrow \overline{S(f,g)}^{\Gamma} = 0$

<u>PR</u> $I = (\underbrace{\underline{\underline{x}}^2-y^3}_{f}, \underbrace{\underline{\underline{x}}^3-y^2}_{g})$ lex

$S(f,g) = \frac{x^3}{x^2}f - \frac{x^3}{x^3}g = -xy^3+y^2$ - nemám čím delit

$I = (\underset{f}{x^2-y^3}, \underset{g}{x^3-y^2}, \underset{h}{xy^3-y^2})$

$S(f,h) = \frac{x^2y^3}{x^2}f - \frac{x^2y^3}{xy^3}\cdot h = -y^6+xy^2$

$I = (x^2-y^3, x^3-y^2, xy^3-y^2, -y^6+xy^2)$

$\vdots$

g lex $I = (y^3-x^2, x^3-y^2)$ - podľa lemy

$Ll(I) = (y^3, x^3)$ $(y^3, x^3) = 1 \Rightarrow$ I - je gröbner. báza

$S(f,g) = -x^5+y^5 : x^3-y^2 = -x^2$

$\underline{-(-x^5+x^2y^2)}$

$-x^2y^2+y^5 : y^3-x^2 = y^2$

$R = k[x_1, \dots x_n]$ $\quad \Gamma = \{f_1, \dots f_s\}$

$f \in R: f = a_1 f_1 + \dots + a_s f_s + r \qquad r = \overline{f}^{\Gamma}$

Veta 1: Ak $\Gamma = \{f_1, \dots f_s\}$ je GRÖBNER. BÁZA ideál $I = (f_1, \dots f_s) \Rightarrow r = \overline{f}$ je určený jednoznačne

Dôkaz. $f = a_1 f_1 + \dots + a_s f_s + r_1 = b_1 f_1 + \dots + b_s f_s + r_2 \Rightarrow r_1 - r_2 \in I \Rightarrow$

$$Ld(r_1 - r_2) \in Ld(I) = (Ld(f_1), \dots, Ld(f_s))$$

lebo je Gröbner. báza

$$\| \quad Ld(r_1) \vee Ld(r_2)$$

$\Rightarrow Ld(r_1) \vee Ld(r_2)$ je deliteľný niektorým $Ld(f_i) \Rightarrow r_1 - r_2 = 0 \Rightarrow$

$$\underline{\underline{r_1 = r_2}}$$

Veta 3: $I = (f_1, \dots f_s)$ G-báza

$f \in k[x_1, \dots x_n] \xrightarrow{\varphi} k[x_1, \dots x_n]/Ld(I)$ epimorfizmus (no)

$f = a_1 f_1 + a_s f_s + \overline{f}^{\Gamma} \longrightarrow \overline{f}^{\Gamma} + Ld(I)$

Nech $f \in \operatorname{Ker} \varphi \Rightarrow \overline{f}^{\Gamma} + Ld(I) = 0 + Ld(I) \Leftrightarrow \overline{f}^{\Gamma} \in Ld(I) \Leftrightarrow$

$\overline{f}^{\Gamma} = 0 \Leftrightarrow f = I$ $\qquad ! \; k[x_1, \dots x_n]/_I \cong k[x_1, \dots x_n]/Ld(I)$

Veta 2: $\Gamma = \{f_1, \dots f_s\}$ je Gröbn. báza ; $f \in I \Leftrightarrow \overline{f}^{\Gamma} = 0$

$\boxed{\Leftarrow}$ zrejmé

$\boxed{\Rightarrow}$ nech $f \in I$ a $\overline{f}^{\Gamma} \neq 0$, keďže $\overline{f}^{\Gamma} \in I \Rightarrow Ld(\overline{f}^{\Gamma}) \in (Ld(f_1) \dots Ld(f_s))$

$\Rightarrow Ld(f_i) / Ld(\overline{f}^{\Gamma})$ $\qquad$ spor

PR: $k[x, y]/(x^2 - y^3, xy - y^2) \cong k[x, y]/(x^2, xy, y^4)$ -báza $1, x, y, y^2, y^3$; rozmer bázy 5

sú izomorfné aj ako vektor. priestory nad k $\quad \| 2$

$k[x, y]/(x^3, xy, y^3)$ $\quad 1, x, x^2, y, y^2$

$I \subset k[x_1, \dots x_n]$ $\qquad \dim I = 0$

$$I = \mathfrak{m}_1 \cap \dots \cap \mathring{\mathfrak{m}}_s$$

$$\updownarrow$$

$$(x_1 - a_1, \dots, x_n - a_n)$$

$$\updownarrow$$

$$e_1 = (a_1, \dots, a_n)$$

$$\| \; \underline{\underline{V(I)}} = \{e_1, \dots e_s\}$$

$$\underline{\underline{\| \; k[x_1, \dots x_n]/_I \xrightarrow{\simeq} k[x_1, \dots, x_n]/_{{}^\circ\mathfrak{m}_1} \oplus k[x_1, \dots x_n]/_{{}^\circ\mathfrak{m}_2} \oplus \dots \oplus k[x_1, \dots, x_n]/_{{}^\circ\mathfrak{m}_s}}}$$

$f \longrightarrow (f + {}^\circ\mathfrak{m}_1, \dots, f + \mathfrak{m}_s)$ $\qquad f \in \operatorname{Ker} \Leftrightarrow f \in {}^\circ\mathfrak{m}_1 \cap \dots \cap {}^\circ\mathfrak{m}_s$

$\Rightarrow \dim_k k[x_1, \dots, x_n]/_I = \sum_{i=1}^{s} \dim_k k[x_1, \dots, x_n]/_{{}^\circ\mathfrak{m}_i}$ // # bodov variety V(I) počítaných s príslušnou násobnosťou

$$\Longrightarrow \dim_k k[x_1,\dots,x_n]/_I = \sum_{i=1}^{r} \dim_k k[x_1,\dots,x_n]/_{\mathcal{O}y_i}$$

$$\overset{\|}{\dim_k} k[x_1,\dots x_n]/_{Ld(I)} \qquad =: \mu(c_1)$$

PR: $(x^3-y^5,\ x-y^4) = I$

$V(I)$ $\quad x^3-y^5=0$
$\quad x-y^4=0$

$y^{12}-y^5=0$
$y^5(y^7-1)=0 \ \begin{cases} y=0 & 5x \\ y^7=1 \end{cases}$ $\quad x=0 \longrightarrow (0,0)\ 5x$

1 $(1,1)$
2 $(\quad)$
3 $(\quad)$
4 $(\quad)$
5 $(\quad)$
6 $(\quad)$
7 $(\quad)$

7 kořenů

$\# V(I) = 12$

lex $\quad (\underbrace{x^3-y^5}_{f},\ \underbrace{x-y^4}_{g}),\ \underbrace{y^{12}-y^5}_{h})$ – Gröbner. báza

$\Longrightarrow Ld(I) = (x, y^{12})$

x^3 nemusím psát, to platí automaticky s x

$$S(f,g) = -y^5 + x^2y^4 : x-y^4 = xy^4+y^8$$
$$\underline{+xy^8 - x^2y^4}$$
$$xy^8 - y^5$$
$$\underline{-xy^8 \mp y^{12}}$$
$$y^{12}-y^5$$

$k[x,y]/_{(x,y^{12})}$

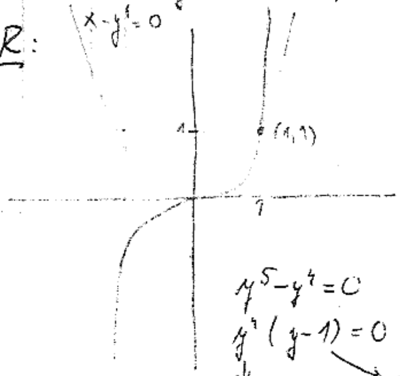
Hledám ∀ monomy, kt. nie sú v ideálu
$1, y, y^2, \dots, y^{11} \longrightarrow \underline{\underline{12}}$

glex
$I = (y^5-x^3, y^4-x),\ x^3-xy)$ $\qquad Ld(I) = (x^3, y^4)$

$S(f,g) = \frac{y^5x}{y^5}(y^5-x^3) - \frac{y^5x}{y^4}(y^4-x) \quad -x^3+xy$ $\qquad k[x,y]/_{(x^2,y^4)}$

$1, x, x^2, y, y^2, y^3, xy, xy^2, xy^3$ [illegible] $\underline{\underline{12}}$

PR:



$I = (x-y^4,\ x-y^5,\ y^5-y^4)$
$S(f,g) = -y^4+y^5$
$Ld(I) = (x, y^5)$ $\quad 1\cdot 5 = 5$
$k[x,y]/_{(x,y^5)}$

$\# V(I) = 5$ $\quad 1, y, y^2, y^3, y^4$ [illegible]

$y^5-y^4=0$
$y^4(y-1)=0$
$\downarrow [0,0]\ 4x \quad \rightarrow [1,1]\ 1x$

PR: $I = (\underbrace{x^2-y^3}_{f},\ \underbrace{x^3-y^2}_{g},\ \underbrace{y^2-z^3}_{h}),\ \underbrace{xy^3-z^3}_{u},\ \underbrace{xz^3-z^9}_{v}$

v glex $(y^2-x^2, x^3-y^2, z^3-y^2)$
$Ld(I) = (x^5, y^3, z^3)$ ↳ gröbnerova báze nesudelitelná

$$S(f,g) = -xy^3+y^2 : y^2-z^3 = 1$$
$$\underline{z^3-y^2}$$
$$\underline{-xy^3+z^3}$$

$S(f,h)$; $S(g,h)$
$\quad\downarrow \qquad\quad \downarrow$
$\quad 0 \qquad\quad 0$

$$S(f,u) = -y^6+xz^3 : y^2-z^3 = -y^4$$
$$\underline{+y^6 \pm y^4z^3}$$
$$xz^3 - y^4z^3 : y^2-z^3 = -y^2z^3$$
$$\underline{\pm y^2z^6 + y^4z^3}$$
$$xz^3 - y^2z^6 : y^2-z^3 = -z^6$$
$$\underline{\mp z^9 + y^2z^6}$$
$$xz^3 - z^9$$

$$S(g,u) \quad -y^5+x^2z^3 : xz^3-z^9 = x$$
$$\underline{xz^9 - x^2z^3}$$
$$xz^9 - y^5 : y^2-z^3 = -y^3$$
$$\underline{y^3z^3 + y^5}$$
$$xz^9 - y^3z^3 : xz^3-z^9 = z^6$$
$$\underline{-xz^9 + z^{15}}$$
$$-y^3z^3 + z^{15} : y^2-z^3 = -yz^3$$
$$\underline{+y^3z^3 \pm z^6y}$$
$$-yz^6 + z^{15}$$
$$\vdots$$

$(F,G) = I$ ak $(Ld(F), Ld(G)) = 1$

nesúdeliteľné, ak (F,G)

$\hookrightarrow$ Gröbner. báza

$a\, Ld(F) + \overline{F} = F$

$b\, Ld(G) + \overline{G} = G$

vedúce ostatné

$$S(F,G) = \frac{Ld(F) \cdot Ld(G)}{a\, Ld(F)} \cdot F - \frac{Ld(F) \cdot Ld(G)}{b\, Ld(G)} \cdot G = \frac{Ld(G)}{a} F - \frac{Ld(F)}{b} G = \underbrace{\frac{Ld(G)}{a}(a\, Ld(F) + \overline{F})}_{} - \underbrace{\frac{Ld(F)}{b} \cdot (}_{}$$

$$(b\, Ld(G) + \overline{G}) = \frac{Ld(G)}{a} \overline{F} - \frac{Ld(F)}{b} \overline{G}$$

$$\frac{Ld(G)}{a} \overline{F} - \frac{Ld(F)}{b} \overline{G} \; : \; b\, Ld(G) + \overline{G} = \frac{1}{ab} \overline{F}$$

$$\underline{-\frac{1}{a} Ld(G) \overline{F} \mp \frac{1}{ab} \overline{F}\,\overline{G}}$$

$$-\frac{Ld(F)}{b} \overline{G} - \frac{1}{ab} \overline{F}\,\overline{G} \; : \; a\, Ld(F) + \overline{F} = -\frac{1}{ab} \cdot \overline{G}$$

$$\underline{+\frac{Ld(F)}{b} \overline{G} + \frac{1}{ab} \overline{F}\,\overline{G}}$$

$$0$$

$\# V(x^a - y^b, x^c - y^d) = \max\{ad, bc\}$

$\# V(x^a - y^b, x^c - z^d, y^e - z^f) = \max\{ade, bcf\}$

TR: $\# \overset{lex}{V}(\underset{f}{x^3 - y^4}, \underset{g}{x^4 - y^3}) = \max\{9, 16\} = 16$

v glex $(\underline{y^4} - x^3, \underline{x^4} - y^3)$ $LL(I) = (x^4, y^4)$ $4 \cdot 4 = 16$

$S(f,g) = -xy^4 + y^3$

$S(f,h) = -y^8 + x^2 y^3$

$V(\underset{f}{x^3 - y^4}, \underset{g}{x^4 - y^3}, \underset{h}{xy^4 - y^3}), x^2 y^3 - y^8$

$\vdots$

LOKÁLNE USPORIADANIA, štandardné bázy

$R := k[x_1, \dots x_n]$

monomiálne usp: lin., komp s náš, $x_i > 1$

lokálne usp:

Df: Usporiadanie $>$ v R nazveme lokálnym, ak platí

1. $>$ je lineárne
2. $>$ je kompatibilné s násobením
3. $1 > x_i \quad \forall i = 1, \dots n$

Def. lex: $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$

$x^\beta = x_1^{\beta_1} \dots x_n^{\beta_1}$ $\alpha - \beta = (0, \dots, 0, \underset{\neq 0}{\alpha_r - \beta_r}, \dots \underset{\neq 0}{\alpha_s - \beta_s}, 0, \dots 0)$

Def: Usporiadanie nazveme reverzne lexikografické ak $\alpha_s - \beta_s < 0$

Keby $x_i > 1 \Rightarrow x_i^2 > x_i$ $\alpha = (0, \dots, 0, 2, \dots 0)$ $\beta = (0, 0, \dots 1, \dots 0)$

$\alpha - \beta = (0, 0, \dots 0, 1, 0 \dots 0)$

lex, glex, grevlex

Každé gradované ... usporiadanie je monomiálne

\+ Antigradované uspor. je lokálne.

↪ antigradované + ...

ANTIGRADOVANÉ LEXIKOGRAFICKÉ USP

Def: usporiadanie > nazveme ↑ ak platí $x^{\alpha} > x^{\beta} \overset{def}{\iff} |\alpha| < |\beta|$ alebo ak $|\alpha| = |\beta|$ potom $\alpha_k - \beta_k > 0$

↪ [illegible] usp [illegible] je [illegible] 0

Def: Bázu $\{F_1, \dots F_s\}$ ideálu $I = (F_1, \dots F_s)$ nazveme štandardnou ak $Ld(I) = (Ld(F_1), \dots, Ld(F_s))$ s určitom lokálnom usporiadaní

\+ platí aj pri Gröbner bázach, ak [illegible] ← Gröbner za štandardnú bázu (menom za lokálnu)

Použitie

$$\begin{tikzcd} I = \mathfrak{m}_1 \arrow[d] & \cap\ \mathfrak{m}_2 \arrow[d] & \cap \dots & \mathfrak{m}_s \arrow[d] \\ (0, 0, \dots 0) & (a_{21}, \dots a_{2n}) & & (a_{s1}, \dots a_{sn}) \end{tikzcd}$$

ak sa [illegible]

$I = (F_1, \dots F_s)$ štandardná

$Ld(I)$ ⇕ platí

$$\dim_k \left(k[x_1, \dots x_n] / Lt(I) \right) = \mu(0)$$

$I = (F_1, \dots F_s)$ Gröbnerova

⇕

$$\dim \left(k[x_1, \dots x_n] / Lt(I) \right) = \# V(I)$$

PR: $I = (x^2 - y^3, x^3 - y^4)$

lex monomiálne ← → antigradované lex = alex

$$S(f,g) = \frac{x^3}{x^2}(x^2 - y^3) - \frac{x^3}{x^3}(x^3 - y^4) = x^3 - xy^3 - x^3 + y^4 = \underline{-xy^3 + y^4}$$

$$I\ (x^2 - y^3, x^3 - y^4, xy^3 - y^4), y^6 - y^5)$$

$$S(f,h) = -y^6 + xy^4 : xy^3 - y^4 = y$$
$$\underline{-(xy^4 - y^5)}$$
$$-y^6 + y^5$$

$$S(g,h) = -y^7 + x^2y^4 : x^2 - y^3 = y^4$$
$$\underline{-y^7 + x^2y^4}$$
$$\emptyset$$

$S(f,u) \to 0$

$S(g,u) \to 0$

$$S(h,u) \quad -y^7 + xy^5 : y^6 - y^5 = -y$$
$$\underline{-(-y^7 - y^9)}$$
$$xy^5 + y^6 : xy^3 - y^4 = y^2$$
$$\underline{-(xy^5 - y^6)}$$

$Ld(I) = (x^2, xy^3, y^6)$

$1, x, y, y^2, y^3, y^4, y^5, xy, xy^2$

$\# V(I) = 9$

alex

$S(f,g) \to 0$
$S(f,h) \to 0$
$S(g,h) \to 0$

$S(f,u) \to 0$
$S(g,u) \to 0$

$$S(h,u) = -y^6 + xy^6 : xy^3 - y^4 = y^3$$
$$-(-y^7 + xy^6)$$
$$-y^6 + y^7 : y^5 - y^6 = -y$$
$$0$$

$Lt(I)$ (x^2, xy^3, y^5)

$1, x, y, y^2, y^3, y^4$ $= 8$
$xy, xy^2,$

$\mu(0) = 8$
$(0,0)$ je [illegible]
[illegible] je [illegible]

Veta : $I = (F_1, \dots F_s)$ Gr. st. báza .

PR: zisti či polynóm je v I $\quad f \in I \Leftrightarrow \overline{f}^{P} = 0$

$$I \,(= x^2 - y^3,\ x^3 - y^4,\ xy^3 - y^4,\ y^6 - y^5)$$

$$f = x^2y - y^4 + x^3y - y^5$$
$$= x^3y + x^2y - y^5 - y^4 : (x^2 - y^3) = xy + y$$
$$-(x^3y - xy^4)$$
$$x^2y + xy^4 - y^5 - y^4$$
$$-(x^2y - y^4)$$
$$xy^4 - y^5 : xy^3 - y^7 = y$$
$$-(xy^4 - y^5)$$
$$0$$

[illegible]obral je len 0 , koľko rôzn. bodov